

Ex A thin uniform circular tube of radius  $a$  contains air and rotates with angular velocity  $\omega$  about an axis in its plane, distance  $c$ , from the centre; find the pressure at any point neglecting the weight of the air. If  $c$  is less than  $a$  and if  $p$  and  $p'$  are the greatest and least pressures, prove that

$$\log \frac{p}{p'} = \frac{\omega^2}{2K} (a+c)^2.$$

Soln

Axis of rotation  $OZ$  is at a distance  $c$  ( $< a$ ) from the centre  $C$  of the circle.

Now consider a point  $P$  at a distance  $r$  from  $OZ$ ; then pressure at this point is given by

$$dp = p \omega r dr. \text{ Also } p = kp.$$

$$\therefore \frac{dp}{p} = \frac{\omega r}{K} dr$$

$$\Rightarrow \log p = \log A + \frac{\omega r^2}{2K}$$

$$\text{or } p = A e^{\frac{\omega r^2}{2K}}$$

This gives pressure at any point of the ~~liquid~~ <sup>air</sup>. Obviously  $p$  will be least when  $r=0$ .

i.e., when  $r=0$ ,  $p=p' \therefore p'=A$ .

Also  $p$  will be the greatest when  $r$  is greatest,

i.e., when  $r=c+a$ , pressure is  $p$ .

$$\therefore p = A e^{\frac{\omega^2}{2K} (a+c)^2} = p' \cdot e^{\frac{\omega^2}{2K} (a+c)^2}$$

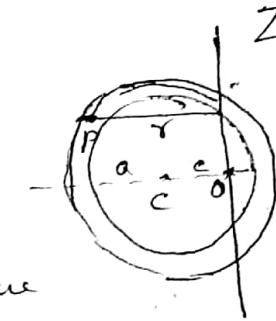
$$\Rightarrow \frac{p}{p'} = e^{\frac{\omega^2}{2K} (a+c)^2}$$

$$\text{or } \log \frac{p}{p'} = \frac{\omega^2}{2K} (a+c)^2$$

This proves the result. /

Ex  
18  
92  
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A bent tube of uniform bore, the arms of which are at right angles, revolves with constant angular velocity  $\omega$  about the axis <sup>one</sup> of its arms, which is vertical and has its extremity immersed in water. Prove that the



height to which the water will rise in the vertical arm is

$$\frac{\pi}{gp} \left(1 - e^{-\frac{w\alpha^2 L}{2k}}\right),$$

$a$  being the length of the horizontal arm,  $\pi$  the atmospheric pressure, and  $p$  the density of water, and  $k$  the ratio of the pressure of the atmosphere to its density.

Sol OA and OB are the arms at right angles to each other. The tube is rotating about the axis of the vertical arm, i.e. about OB.

Take a point P in OA at a distance  $r$  from O; then pressure of the air at this point is given by

$$dp = \rho \omega r dr \quad \text{where } p = kp.$$

$$\text{or } \frac{dp}{p} = \frac{\omega r}{k} dr$$

$$\text{Integrating, } p = C e^{\frac{\omega r^2}{2k}}$$

At A where  $OA = a$ ,  $p = \pi$ , the atmospheric pressure, as the end A is open.

$$\therefore \text{when } r = a, p = \pi; \therefore \pi = C e^{\frac{\omega a^2}{2k}}$$

$$\therefore p = \pi e^{\frac{\omega r^2}{2k}(r-a^2)}$$

putting  $r=0$ , the pressure at O is given by

$$p = \pi e^{-\frac{\omega a^2}{2k}}$$

This is pressure in the vertical arm.

Now, if liquid in the vertical tube stands to a height  $h (= EB)$ , then since the pressure at B is  $\pi$ , we have

$$\rho gh + \pi e^{-\frac{\omega a^2 L}{2k}} = \pi$$

$$\Rightarrow h = \frac{\pi}{\rho g} \left(1 - e^{-\frac{\omega a^2 L}{2k}}\right). \quad \text{This proves the result.}$$

Ex 107 Taking into account the variation of gravity with height and assuming that the temperature of the air is constant at all heights, prove that at height  $x$ , the pressure  $p$  of the air is given by

$$\log \frac{p}{p_0} = - \frac{g_0 ax}{K(a+x)},$$

where  $a$  is the earth's radius,  $K = \frac{p_0}{\rho_0}$  and  $p_0, \rho_0, g_0$  are the values of the pressure, density and gravity at the earth's surface.

Soln A point at a distance  $x$  from the earth is at a distance  $a+x$  from the centre of the earth; therefore if  $g$  is the attraction due to gravity at the point, then

$g = \frac{\lambda}{(a+x)^r}$ . Also  $g_0 = \frac{\lambda}{a^r}$  at the earth's surface.

$$\therefore g = \frac{g_0 a^r}{(a+x)^r}.$$

Hence the equation of the pressure is

$$dp = -g\rho dx,$$

$$\text{i.e., } dp = - \frac{g_0 a^r}{(a+x)^r} \rho dx.$$

Also  $\rho = kp$  ( $\because$  temperature is constant).

$$\text{or } \frac{dp}{p} = - \frac{g_0 a^r}{K(a+x)^r} dx.$$

$$\text{Integrating, } \log p = \frac{g_0 a^r}{K(a+x)} + C$$

At earth's surface,  $x=0, p=p_0$ .

$$\therefore \log p_0 = \frac{g_0 a^r}{Ka} + C.$$

$$\therefore \log \frac{p}{p_0} = - \frac{g_0 ax}{K(a+x)}$$

This proves the result.

Ex ✓ A hollow gas-tight sphere, containing hydrogen, requires a force  $mg$  to prevent it from rising when its lowest point touches the ground; the total mass of the sphere and hydrogen is  $M$ . Show that the sphere can float in equilibrium with its lowest point at a height  $h$  above the ground, where

$$h = \frac{K}{g} \log \frac{M+m}{M},$$

$K$  being the ratio of the pressure of the atmosphere to its density.

Soln. At a height  $z$ , pressure of the air is given by

$$dp = -\rho g dz, \text{ where } \rho = kp,$$

$$\therefore \frac{dp}{p} = -\frac{g}{k} dz.$$

$$\text{or } \log p = -\frac{g}{k} z + C$$

$$\text{or } p = p_0 e^{-\frac{gz}{k}}$$

But when  $z=0$ ,  $p=p_0$ ;  $\therefore C=p_0$ .

$$\text{Thus, } p = p_0 e^{-\frac{gz}{k}} \quad \text{--- (1)}$$

If  $\rho_1$  is the density, when  $z=h$ , i.e. at a height  $h$ ,

$$\text{then } \rho_1 = p_0 e^{-\frac{gh}{k}}. \quad \text{--- (2)}$$

Now, let  $V$  be the volume of the sphere,  $\rho_0$  and  $\rho_1$  the densities at bottom ( $z=0$ ) and at a height  $h$  [ $z=h$ ], given by (2)]

Since a weight  $mg$  is required to keep the sphere at the bottom, we have

$$Mg + mg = V\rho_0 g. \quad \text{--- (3)}$$

If it floats in equilibrium at a height  $h$ , then in this position,  $Mg = V\rho_1 g$ .  $\quad \text{--- (4)}$

$$\text{So that } \frac{p_0}{\rho_1} = \frac{M+m}{M}$$

$$\therefore \frac{P_0}{P_0 e^{-gR/k}} = \frac{M+m}{M}$$

$$\therefore e^{gR/k} = \frac{M+m}{M}$$

$$\text{or } R = \frac{k}{g} \log \frac{M+m}{M} \quad \text{H.}$$

## Mixtures of Gases:

Proposition: If two gases contained in two vessels of volume  $V$  each have the same temperature but pressures  $p_1$  and  $p_2$  be mixed so that the volume of the mixture is  $V$ , then the pressure of the mixture so obtained is  $p_1 + p_2$ .

Pf. Let the gas, of which the pressure is  $p_1$ , have its ~~and pressure~~ volume changed, keeping the temperature constant, to  $p_2$ . If  $V'$  is the volume now, then by Boyle's law,

$$p_2 V' = p_1 V \quad \text{or} \quad V' = \frac{p_1 V}{p_2}$$

now let the two gases be mixed (now the two gases are at the same temperature and pressure  $p_2$ ), then ~~by~~ the ~~above~~ volume of the mixture

$$= V + V' = V + \frac{p_1 V}{p_2} = \frac{p_1 + p_2}{p_2} V$$

But the volume of the mixture is to be reduced to  $V$ ; hence when  $V$  is the volume of the mixture, let  $p$  be its pressure; then by Boyle's law for this mixture now, we have

$$pV = \left( \frac{p_1 + p_2}{p_2} V \right) p_2$$

$$\Rightarrow p = p_1 + p_2$$

proved.

Thermal Capacity: The thermal capacity of a body is the amount of heat required to raise its temperature by one degree.

Specific heat: Specific heat of a body is the ratio of the amount of heat required to increase by  $1^{\circ}\text{C}$  the temperature of the body to the amount of heat required to increase by  $1^{\circ}\text{C}$  the temperature of an equal weight of water.

Specific heat may also be defined as thermal capacity of a unit mass of the body.

Let an amount of heat  $dQ$  produce a change  $dt$  in the temperature of a unit mass; then  
specific heat =  $\frac{dQ}{dt}$ .

In case of a gas, the temperature can be changed in two ways:

- by keeping pressure constant,
- by keeping volume constant.

The specific heat when the pressure is kept constant is denoted by  $C_p$  and when volume is kept constant, it is denoted by  $C_v$ .

Internal Energy: Internal energy of a body is the energy which is required to create the body or the system. [Or, the energy in a system arising from the relative positions and interactions of its parts.]

Prove that  $C_p$  is greater than  $C_v$  for a perfect gas.

If a quantity  $dQ$  of heat cause an increase  $dE$  in the internal energy and an increase  $dV$  in volume against a pressure  $P$ , then the first law of thermodynamics may be expressed as

$$dQ = dE + pdV \quad \text{--- (1)}$$

117 Perfect gas: A perfect gas is an ideal substance which obeys the relation  $PV = RT$  for all ranges of temperature where  $T$  denotes the absolute temperature and  $R$  is a constant.

By experiment it is obtained that  $dQ$  is a function of  $T$  for a perfect gas.

First let the volume be kept constant while a quantity  $dQ$  of heat is imparted.

$\therefore V$  is constant,  $\therefore dV = 0$ .

$$\text{Then (1)} \Rightarrow dQ = dE$$

Also when  $V$  is kept constant, we have

$$\frac{dQ}{dT} = C_V \quad \text{or} \quad dQ = C_V dT$$

$$\therefore dE = C_V dT$$

$$\therefore (1) \Rightarrow dQ = C_V dT + pdV \quad \text{--- (2)}$$

Also for a perfect gas,

$$PV = RT$$

$$\Rightarrow pdV + Vdp = RdT$$

$$\Rightarrow pdV = RdT - Vdp \quad \text{--- (3)}$$

From (2) and (3),

$$dQ = C_V dT + RdT - Vdp \quad \text{--- (4)}$$

Now, suppose the pressure be kept constant while a quantity  $dQ$  of heat is imparted; then

$$\frac{dQ}{dT} = C_P$$

$$\Rightarrow dQ = C_P dT \quad \text{and} \quad dp = 0 \quad \text{--- (5)}$$

$$\therefore (4) \text{ and (5)} \Rightarrow C_P dT = C_V dT + RdT - 0$$

$$\Rightarrow C_P = C_V + R$$

$$\Rightarrow C_P - C_V = R \Rightarrow C_P > C_V. \quad \underline{\text{proved}}$$

## XII Adiabatic Expansion

( $a \equiv$  not,  $dq =$  though,  $bates =$  heat  $\therefore$  no heat passing through).

Let there be a change in the state of the gas in such a way that no heat is lost or gained by the gas; then such an expansion or compression is called an adiabatic change.

Ques: For an adiabatic change, prove the relation  
116  $PV^r = \text{constant}$ .

PF-

~~∴ There is no heat lost or gained,~~

$$\cancel{dq = 0} \quad \rightarrow 0$$

Also  $\cancel{dq = dE + pdv - ①}$  [First law of thermo dynamics]  
where a quantity of heat  $dq$  causes an increase  $dE$  in the internal energy and an increase  $dv$  in volume against a pressure  $p$ .

$$\cancel{\therefore ① + ② \Rightarrow 0 = dE + pdv - ③}$$

Also, when  $v$  is kept constant, then  $dv = 0$   
then  $\cancel{③ \Rightarrow dq = dE}$ .  
Also when  $v$  is kept constant  $\frac{dq}{dT} = C_V$

$$\Rightarrow dq = C_V dT \quad \text{and } dv = 0.$$

$$\therefore \cancel{③ \Rightarrow dq = dE}$$

$$\therefore \cancel{① \Rightarrow dq = C_V dT + pdv}$$

PF-

If a quantity  $dq$  of heat increases  $\cancel{dq}$  causes an  $dE$  in the internal energy, and an increase  $dv$  in volume against a pressure  $p$ , then by first ~~to~~ law of thermodynamics,

$$\text{we get } dQ = dE + pdv \quad \text{--- (1)}$$

Let the volume  $v$  be kept constant.

Then  $dv = 0$ .

$$\therefore (1) \Rightarrow dQ = dE$$

$$\text{Also, then } \frac{dQ}{dT} = C_V \\ \Rightarrow dQ = C_V dT$$

$$\therefore (1) \Rightarrow dQ = C_V dT + pdv \quad \text{--- (2)}$$

Now, if ~~there is~~ no heat is lost or gained

(i.e. adiabatic change), then  $dQ = 0$ .

$$\therefore (2) \Rightarrow 0 = C_V dT + pdv \quad \text{--- (3)}$$

Also for a perfect gas,  $pV = RT$  and  $C_p - C_V = R$ .

$$\therefore pdv + vdp = RdT \\ = (C_p - C_V) dT \quad \text{--- (4)}$$

Eliminating  $dT$  from (3) and (4),

we get -

$$pdv + vdp = (C_p - C_V) \cdot \left(-\frac{p}{C_p} dv\right)$$

$$= \left(-\frac{C_p}{C_V} + 1\right) pdv$$

$$\Rightarrow vdp = p \left(1 - \frac{C_p}{C_V} - 1\right) dv$$

$$\Rightarrow \frac{dp}{p} = -\frac{C_p}{C_V} \frac{dv}{v}$$

$$\Rightarrow \frac{dp}{p} + \gamma \frac{dv}{v} = 0 \quad \text{where } \frac{C_p}{C_V} = \gamma$$

Integrating  $\log p + \gamma \log v = \log (\text{const})$ , ~~= const~~

$$\Rightarrow pV^\gamma = \text{Const.}$$

proved